

WOLFF'S THEOREM ON IDEALS FOR MATRICES

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ABSTRACT. We extend Wolff's theorem concerning ideals on $H^\infty(\mathbb{D})$ to the matrix case, giving conditions under which an H^∞ -solution G to the equation $FG = H$ exists for all $z \in \mathbb{D}$, where F is an $m \times \infty$ matrix of functions in $H^\infty(\mathbb{D})$, and H is an $m \times 1$ vector of such functions. We then examine several useful results.

1. INTRODUCTION

In 1962, Lennart Carleson [3] solved the Corona Problem, proving that the ideal \mathcal{I} generated by a finite set of functions $\{f_i\}_{i=1}^n \subset H^\infty(\mathbb{D})$ is the entire space $H^\infty(\mathbb{D})$ provided there exists $\delta > 0$ such that

$$(1) \quad \sum_{i=1}^n |f_i(z)| \geq \delta \quad \forall z \in \mathbb{D}.$$

This result can be extended to hold for infinitely many functions $\{f_i\}_{i=1}^\infty$ (see [8], [13]). Two different extensions of the corona theorem to matrices were given by Fuhrmann [5] and Andersson [1]. Fuhrmann's result was extended to one-sided infinite matrices by Vasyunin (see Nikolski [7]). However, Treil [10] showed that a complete extension of the corona theorem is not possible in the two-sided infinite matrix case. Trent and Zhang proved that the result of Furmann and Vasyunin can be extended to any algebra that satisfies a corona theorem [15], and later did the same for Andersson's result, also allowing for one-sided infinite matrices [16].

A more general question than Carleson's is, under what conditions is a given function $h \in H^\infty(\mathbb{D})$ to be found in \mathcal{I} ? One might suppose based on Carleson's result that a sufficient condition would be that

$$(2) \quad \sum_{i=1}^n |f_i(z)| \geq |h(z)| \quad \forall z \in \mathbb{D},$$

but that is not the case, as Rao proved (see Garnett [6]). Thomas Wolff, however, proved that, given (2), $h^3 \in \mathcal{I}$ [17]. More recently, Treil showed that the result fails when the exponent "3" is replaced with "2" (although it holds for any exponent greater than 2) [11].

If we adjust the hypothesis to

$$(3) \quad \left[\sum_{i=1}^n |f_i(z)|^2 \right]^{\frac{3}{2}} \geq |h(z)| \quad \forall z \in \mathbb{D},$$

we obtain $h \in \mathcal{I}$. A matter of current interest is how this estimate can be improved, which we will discuss at the end of this paper. For now, (3) will be sufficient for our needs.

Since the corona theorem has been applied to matrices, a natural question is whether the same can be done for Wolff's theorem. The answer is yes, as we will now set forth to prove. First, however, a definition is in order.

Definition 1.1. Let $B \in M_m(\mathbb{C})$. For $1 \leq k \leq m$, define

$$\det_k(B) = \sum_{\pi \in \Pi_k(m)} \det(E_\pi B E_\pi)$$

where $\Pi_k(m)$ denotes the increasing k -tuples of integers in $\{1, 2, \dots, m\}$.

Here E_π is the $m \times m$ matrix whose i th column is the i th column of the $m \times m$ identity matrix if $i \in \pi$, and is zero otherwise. When taking the determinant of $E_\pi B E_\pi$ in the above definition, we delete those columns and rows consisting of all zeros.

Wolff's Theorem for Matrices. Let $F(z)$ be an $m \times \infty$ matrix of functions in $H^\infty(\mathbb{D})$ with $\max\{\text{rank } F(z) \mid z \in \mathbb{D}\} = k \leq m$. Let $H(z)$ be an $m \times 1$ vector of functions in $H^\infty(\mathbb{D})$. Suppose

- (i) $[\det_k(F(z)F(z)^*)]^{\frac{3}{2}} \geq |h_i(z)| \forall z \in \mathbb{D}, i = 1, \dots, m$
- (ii) $\|M_F\| = 1$
- (iii) there exists a function $\underline{u} : \mathbb{D} \rightarrow l^2$ such that $F\underline{u} = H$ everywhere on \mathbb{D} .

Then there exists an $\infty \times 1$ vector $G(z)$ of functions in $H^\infty(\mathbb{D})$ such that

- (a) $F(z)G(z) = H(z) \forall z \in \mathbb{D}$, and
- (b) $\|M_G\| < \infty$.

We base our arguments on those found in Trent and Zhang [16]. The main difference here is that we do not assume a uniform lower boundedness on F , and instead assume that F is bounded by the entries of H .

2. PRELIMINARIES

Before giving the proof of Wolff's Theorem for Matrices, we define and list some properties of "Q-operators." Proofs of these properties can be found in [15].

We let $H \wedge K$ denote the exterior product between two Hilbert spaces H and K , and $l_{(n)}^2 = \wedge_{i=1}^n l^2$. In keeping with this notation, $l_{(0)}^2 = \mathbb{C}$.

Let $\{e_i\}_{i=1}^\infty$ denote the standard basis in l^2 . If I_n denotes increasing n -tuples of positive integers and if $(i_1, i_2, \dots, i_n) \in I_n$, we let $\pi_n = \{i_1, i_2, \dots, i_n\}$ and, abusing notation, we write $\pi \in I_n$. If we define $e_{\pi_n} = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n}$, then $\{e_{\pi_n}\}_{\pi_n \in I_n}$ is defined to be the standard basis for $l_{(n)}^2$.

Let $H(E)$ be a reproducing kernel Hilbert space on a set E , and let $\mathcal{A} = M(H(E))$, the multiplier algebra on $H(E)$. Let $\underline{f}_j(z) = (v_1(z), v_2(z), \dots)$, where $\{v_n\}_{n=1}^\infty \subset \mathcal{A}$, such that $\underline{f}_j(z)\underline{f}_j(z)^* \leq 1 \forall z \in E$. Fix $z \in E$, and for $n = 0, 1, \dots$ define

$$Q_j^{(n)*}(z) : l_{(n)}^2 \rightarrow l_{(n+1)}^2$$

by

$$Q_j^{(n)*}(z)(w_n) = \overline{\underline{f}_j(z)} \wedge \underline{w}_n,$$

where $\underline{w}_n \in l_{(n)}^2$. Note that $Q_j^{(0)*}(z) = \overline{\underline{f}_j(z)}$.

We observe that $\text{ran } Q_j^{(n)*}(z) \subset \ker Q_j^{(n+1)*}(z)$. Furthermore, equality can be shown if we stipulate that $\underline{f}_j(z)\underline{f}_j(z)^* \geq \delta > 0$. (This follows from (5) below.)

By the anti-commutivity of the exterior product, we see that

$$(4) \quad Q_j^{(n)} Q_k^{(n+1)} = -Q_k^{(n)} Q_j^{(n+1)}.$$

Also, for $e_{\pi_n} \in l_{(n)}^2$, we have

$$Q_j^{(n)*}(z)(e_{\pi_n}) = \overline{f_j(z)} \wedge e_{\pi_n} = \left(\sum_{p=1}^{\infty} \overline{v_p(z)} e_p \right) \wedge e_{\pi_n},$$

so with respect to the standard basis, the entries in $Q_j^{(n)*}(z)$ are 0 or $\pm \overline{v_n(z)}$ for some n . Thus $Q_j^{(n)}(\cdot)$ has entries belonging to \mathcal{A} with respect to the standard basis.

Assume that there exists a fixed $z \in E$ such that $f_j(z) \neq 0$, and let $Q_n = Q_j^{(n)}(z)$. Then for $n = 0, 1, \dots$,

$$(5) \quad Q_n^* Q_n + Q_{n+1} Q_{n+1}^* = \|a\|^2 I_{l_{n+1}^2}.$$

Finally, if $a_i = \underline{f}_i(z)$, then for $k = 2, \dots, m$, let $a'_k = P_{\text{sp}\{a_1, \dots, a_{k-1}\}}^{\perp}(a_k)$. Then

$$(6) \quad \begin{aligned} a_1 Q_{a_2}^{(1)} \dots Q_{a_m}^{(m-1)} Q_{a_m}^{(m-1)*} \dots Q_{a_2}^{(1)*} a_1^* &= \|a_1\|^2 \prod_{j=2}^m \|a'_j\|^2 \\ &= \det(F(z)F(z)^*). \end{aligned}$$

The last equality is obtained by a straightforward computation, using the fact that $\frac{a_i^* a_i}{\|a_i\|^2}$ is the rank one projection of l^2 onto a_i .

We obtain a very nice norm estimate in the case where $\mathcal{A} = H^\infty(\mathbb{D})$. Let $A = (a_1, a_2, \dots)$, $a_i \in H^\infty(\mathbb{D})$, with $\|M_A\| < \infty$. From (5) we have the following estimate on the operator norms (for fixed $z \in \mathbb{D}$):

$$\|Q^{(j)}\| \leq \|A\|.$$

Now

$$\|M_{Q^{(j)}}\| = \sup_{z \in \mathbb{D}} \|Q^{(j)}(z)\| \leq \sup_{z \in \mathbb{D}} \|A(z)\| = \|M_A\|.$$

(Here $j \geq 1$. If $j = 0$, then $Q^{(j)} = A$, so the result is trivial.) Thus

$$(7) \quad \|M_{Q^{(j)}}\| \leq \|M_A\|.$$

We will need Lemma 1 from [16] if we wish to extend our main theorem to spaces besides $H^\infty(\mathbb{D})$, however.

The following lemmas will be used in the proof of our theorem. Their proofs have been omitted but may be found in [16].

Definition 2.1. Let $\{X_j\}_{j=1}^{n+1}$ be Banach spaces and let $\{T_{jk}\}_{j,k=1}^n$ denote operators such that $T_{jk} \in B(X_{j+1}, X_j)$. Let

$$\text{“det”} \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \vdots & & & \vdots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{pmatrix} = \sum_{\substack{\sigma \in P(n) \\ \sigma = \{i_1, \dots, i_n\}}} (-1)^{\text{sgn}(\sigma)} T_{1i_1} T_{2i_2} \dots T_{ni_n}$$

where the products are given in the order indicated and $P(n)$ denotes the permutations of $\{1, \dots, n\}$ and $\text{sgn}(\sigma)$ denotes the sign of the permutation of σ .

Lemma 2.1.

$$\begin{aligned}
 & \text{“det”} \begin{pmatrix} H_1 & H_{i_1} & \dots & H_{i_p} \\ \underline{f}_1 & \underline{f}_{i_1} & \dots & \underline{f}_{i_p} \\ Q_1^{(1)} & Q_{i_1}^{(1)} & \dots & Q_{i_p}^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ Q_1^{(p-1)} & Q_{i_1}^{(p-1)} & \dots & Q_{i_p}^{(p-1)} \end{pmatrix} \\
 &= p! [H_1 \underline{f}_{i_1} Q_{i_2}^{(1)} \dots Q_{i_p}^{(p-1)} + \sum_{l=1}^p (-1)^l \underline{f}_1 H_{i_l} Q_{i_1}^{(1)} \dots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \dots Q_{i_p}^{(p-1)}].
 \end{aligned}$$

Lemma 2.2. *Suppose $F(\cdot)$ is $m \times \infty$ with $F(z)F(z)^*$ having maximum rank $p < m$. Suppose that for some function $\underline{u} : E \rightarrow l^2$, $F(z)\underline{u}(z) = H(z)$, where $H = (h_1, \dots, h_m)$ with $h_j \in M(H(E))$. Then for any $\pi \in \Pi_{(p+1)}(n)$ with $\pi = (j_1, \dots, j_{p+1})$, we have for each $z \in E$*

$$\text{“det”} \begin{pmatrix} H_{i_1} & \dots & H_{i_{p+1}} \\ \underline{f}_{i_1} & \dots & \underline{f}_{i_{p+1}} \\ Q_{i_1}^{(1)} & \dots & Q_{i_{p+1}}^{(1)} \\ \vdots & \vdots & \vdots \\ Q_{i_1}^{(p-1)} & \dots & Q_{i_{p+1}}^{(p-1)} \end{pmatrix} = 0.$$

3. PROOF OF WOLFF’S THEOREM FOR MATRICES

We are now ready to prove our theorem.

Proof. The solution G that we seek can be written as a sum of vectors G_1, \dots, G_m . We will find the vector G_1 here; vectors G_2, \dots, G_m are found similarly.

Let $k \leq m$. By the same process we used to obtain (6), we see that

$$\left[\sum_{\substack{\pi \in \Pi_k(m) \\ \pi = \{i_1, \dots, i_k\}}} \underline{f}_{i_1}(z) Q_{i_2}^{(1)}(z) \dots Q_{i_k}^{(k-1)}(z) Q_{i_k}^{(k-1)*}(z) \dots Q_{i_2}^{(1)*}(z) \underline{f}_{i_1}^*(z) \right]^{\frac{3}{2}} \geq |h_i(z)|$$

for all $z \in \mathbb{D}$ and $i = 1, \dots, m$. Using (7),

$$\| M_{\underline{f}_{i_1}} M_{Q_{i_2}^{(1)}} \dots M_{Q_{i_k}^{(k-1)}} \| < \infty,$$

so by Wolff’s Theorem there exists, for each $\pi \in \Pi_k(m)$, a vector v_π with entries in $H^\infty(\mathbb{D})$ such that

$$k! \sum_{\substack{\pi \in \Pi_k(m) \\ \pi = \{i_1, \dots, i_k\}}} \underline{f}_{i_1} Q_{i_2}^{(1)} \dots Q_{i_k}^{(k-1)} v_\pi = h_1$$

and

$$\| M_{v_\pi} \| < \infty.$$

We can rewrite this equation in terms of exterior algebras as

$$\begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_m \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \dots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \cdot v_1 = h_1$$

where v_1 is a vector with $\binom{m}{k}$ entries v_π for $\pi \in \Pi_k(m)$. Then we have $\|M_{v_i}\| < \infty$.

We claim the vector

$$G_1 = k \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ Q_2^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \cdot v_1$$

is the vector we seek. To prove this, we will consider a more general vector,

$$A = k \begin{pmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_m I \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ Q_2^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix}$$

where $\alpha_1, \dots, \alpha_m \in H^\infty(\mathbb{D})$. Now

$$\begin{aligned} \underline{f}_1 A &= k \begin{pmatrix} \alpha_1 \underline{f}_1 \\ \vdots \\ \alpha_m \underline{f}_1 \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \\ &= k! \begin{pmatrix} \alpha_1 \underline{f}_1 \\ \vdots \\ \alpha_m \underline{f}_1 \end{pmatrix} \wedge \sum_{\substack{\sigma \in \Pi_{k-1}(m) \\ \sigma = (j_2, \dots, j_k)}} Q_{j_2}^{(1)} \cdots Q_{j_k}^{(k-1)} e_\sigma \\ &= k! \sum_{j=1}^m \sum_{\substack{\sigma \in \Pi_{k-1}(m) \\ \sigma = (j_2, \dots, j_k) \\ 1 \notin \sigma}} \alpha_j \underline{f}_1 Q_{j_2}^{(1)} \cdots Q_{j_k}^{(k-1)} e_j \wedge e_\sigma \\ &= k! \alpha_1 \sum_{\substack{\sigma \in \Pi_{k-1}(m) \\ \sigma = (j_2, \dots, j_k) \\ 1 \notin \sigma}} \underline{f}_1 Q_{j_2}^{(1)} \cdots Q_{j_k}^{(k-1)} e_1 \wedge e_\sigma \\ &\quad + k! \sum_{j=2}^m \sum_{\substack{\sigma \in \Pi_{k-1}(m) \\ \sigma = (j_2, \dots, j_k) \\ 1, j \notin \sigma}} \alpha_j \underline{f}_1 Q_{j_2}^{(1)} \cdots Q_{j_k}^{(k-1)} e_j \wedge e_\sigma \\ &= k! \alpha_1 \sum_{\substack{\pi \in \Pi_k(m) \\ \pi = (1, i_2, \dots, i_k)}} \underline{f}_1 Q_{i_2}^{(1)} \cdots Q_{i_k}^{(k-1)} e_\pi \\ &\quad + k! \sum_{\substack{\pi \in \Pi_k(m) \\ \pi = (i_1, \dots, i_k) \\ 1 \notin \pi}} \sum_{l=1}^k (-1)^{l-1} \alpha_{i_l} \underline{f}_1 Q_{i_1}^{(1)} \cdots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \cdots Q_{i_k}^{(k-1)} e_\pi. \end{aligned}$$

For the second and third equalities, we simply applied the definition of the exterior product. For the fourth, we broke the summation into two parts. The last inequality is obtained by renaming the indices and using (4).

By Lemma 2.1,

$$\begin{aligned} & k! \sum_{l=1}^k (-1)^l \alpha_{i_l} \underline{f}_1 Q_{i_1}^{(1)} \dots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \dots Q_{i_k}^{(k-1)} \\ &= \text{“det”} \begin{pmatrix} \alpha_1 & \alpha_{i_1} & \dots & \alpha_{i_k} \\ \underline{f}_1 & \underline{f}_{i_1} & \dots & \underline{f}_{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ Q_1^{(k-1)} & Q_{i_1}^{(k-1)} & \dots & Q_{i_k}^{(k-1)} \end{pmatrix} - k! \alpha_1 \underline{f}_{i_1} Q_{i_2}^{(1)} \dots Q_{i_k}^{(k-1)}. \end{aligned}$$

But $\text{rank } F(z)F(z)^* = k$, so by Lemma 2.2, the “det” term equals 0. Thus

$$k! \sum_{l=1}^k (-1)^{l-1} \alpha_{i_l} \underline{f}_1 Q_{i_1}^{(1)} \dots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \dots Q_{i_k}^{(k-1)} = k! \alpha_1 \underline{f}_{i_1} Q_{i_2}^{(1)} \dots Q_{i_k}^{(k-1)}.$$

Now we have

$$\begin{aligned} \underline{f}_1 A &= k! \alpha_1 \sum_{\substack{\pi \in \Pi_k(m) \\ \pi = (1, i_2, \dots, i_k)}} \underline{f}_1 Q_{i_2}^{(1)} \dots Q_{i_k}^{(k-1)} e_\pi \\ &+ k! \alpha_1 \sum_{\substack{\pi \in \Pi_k(m) \\ \pi = (i_1, i_2, \dots, i_k) \\ 1 \notin \pi}} \underline{f}_{i_1} Q_{i_2}^{(1)} \dots Q_{i_k}^{(k-1)} e_\pi \\ &= k! \alpha_1 \sum_{\substack{\pi \in \Pi_k(m) \\ \pi = (i_1, i_2, \dots, i_k)}} \underline{f}_{i_1} Q_{i_2}^{(1)} \dots Q_{i_k}^{(k-1)} e_\pi \\ &= \alpha_1 \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_m \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \dots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \underline{f}_1 G_1 &= 1 \cdot \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_m \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \dots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \cdot v_1 \\ &= h_1 \end{aligned}$$

and similarly,

$$\begin{aligned} \underline{f}_i G_1 &= 0 \cdot \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_m \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \dots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \cdot v_1 \\ &= 0 \end{aligned}$$

for $i \neq 1$.

We need only show that $\|M_G\| < \infty$. By (7),

$$\begin{aligned}
\|M_{G_1}\| &= \|k \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ Q_2^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \cdot v_1\| \\
&\leq k \left\| \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ Q_2^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \right\| \|M_{v_1}\| \\
&= k! \left\| \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \sum_{\substack{\pi \in \Pi_k(m) \\ \pi=1, i_2, \dots, i_k}} Q_{i_2}^{(1)} \cdots Q_{i_k}^{(k-1)} \right\| \|M_{v_1}\| \\
&= k! \left\| \sum_{\substack{\pi \in \Pi_k(m) \\ \pi=1, i_2, \dots, i_k}} Q_{i_2}^{(1)} \cdots Q_{i_k}^{(k-1)} \right\| \|M_{v_1}\| \\
&\leq k! \left(\sum_{\substack{\pi \in \Pi_k(m) \\ \pi=1, i_2, \dots, i_k}} \|M_{Q_{i_2}^{(1)}}\| \cdots \|M_{Q_{i_k}^{(k-1)}}\| \right) \|M_{v_1}\| \\
&\leq k! \left(\sum_{\substack{\pi \in \Pi_k(m) \\ \pi=1, i_2, \dots, i_k}} \|M_{f_{i_2}}\| \cdots \|M_{f_{i_k}}\| \right) \|M_{v_1}\| \\
&\leq k! \sum_{\substack{\pi \in \Pi_k(m) \\ \pi=1, i_2, \dots, i_k}} \|M_{v_1}\| \\
&= k! \binom{m-1}{k-1} \|M_{v_1}\|.
\end{aligned}$$

Using the estimates in [14] for $\alpha(t) = t^{\frac{1}{2}}$, we obtain

$$\|M_{v_1}\| \leq 1 + 4\sqrt{e} + 8\sqrt{2}e + 72e^{\frac{3}{2}} = K < 362$$

so

$$\|M_G\| \leq mk! \binom{m-1}{k-1} \|M_{v_1}\| \leq mk! \binom{m-1}{k-1} \frac{K}{k!} \leq m \binom{m-1}{k-1} K.$$

This concludes our proof. \square

4. FURTHER RESULTS

4.1. Improved Estimates. As noted in the introduction, one can improve the estimate in Wolff's theorem. The exponent " $\frac{3}{2}$ " used in our hypotheses isn't optimal, but was used for convenience.

Cegrell [4] showed that (3) can be replaced with

$$(8) \quad F(z)F(z)^*\alpha(F(z)F(z)^*) \geq |h(z)| \quad \forall z \in \mathbb{D}$$

where

$$(9) \quad \alpha(t) = A_0(\ln \frac{c}{t})^{-\frac{3}{2}}(\ln \ln \frac{c}{t})^{-\frac{3}{2}}(\ln \ln \ln \frac{c}{t})^{-1}$$

for $t \in (0, 1]$ and $\alpha(0) = 0$, and for $F(z) = (f_1(z), \dots, f_n(z))$. Here c is chosen so that all log expressions are positive, and A_0 is chosen so that $\alpha(1) = 1$. Trent [14] improved on this estimate (and also allowed for infinitely many functions f_i). The best estimate is currently due to Treil [12]. These and further improvements in the estimate automatically carry over to our theorem.

4.2. Extensions to Other Spaces. Although we restricted our attention to functions in $H^\infty(\mathbb{D})$, the methods used in the proof of Wolff's theorem for matrices apply to any algebra of functions that satisfies a Wolff theorem. (Note that some hypotheses may have to be changed. For example, on $H^\infty(\mathbb{D})$, $\|M_F\| = \|M_F^T\|$, but on other spaces we may have to stipulate that $\max\{\|M_F\|, \|M_F^T\|\} < \infty$.)

As an example, consider Dirichlet space, $\mathcal{D}^2(\mathbb{D})$, defined by

$$\mathcal{D}^2(\mathbb{D}) = \{f : \mathbb{D} \rightarrow \mathbb{C} \mid f \text{ is analytic on } \mathbb{D}, \\ f(z) = \sum_{n=0}^{\infty} a_n z^n, \|f\|^2 = \sum_{n=0}^{\infty} (n+1)|a_n|^2 < \infty\}.$$

Banjade [2] has recently proved that the algebra of multipliers on Dirichlet space, $M(\mathcal{D}^2(\mathbb{D}))$, satisfies a Wolff theorem: given $\{f_j\}_{j=1}^{\infty} \subset M(\mathcal{D}^2(\mathbb{D}))$ and $h \in M(\mathcal{D}^2(\mathbb{D}))$ such that (8) holds,

$$|F'(z)F^*(z)|\alpha(F(z)F(z)^*) \geq |h'(z)| \quad \forall z \in \mathbb{D}$$

and

$$\|M_F\| \leq 1,$$

where $F(z) = (f_1(z), f_2(z), \dots)$ and α is as in (9), then there exists $\{g_j\}_{j=1}^{\infty} \subset M(\mathcal{D}^2(\mathbb{D}))$ with $G(z) = (g_1(z), g_2(z), \dots)$ such that $F(z)G(z)^T = h(z) \quad \forall z \in \mathbb{D}$, and $\|M_G\| < \infty$.

Thus, given an $m \times \infty$ matrix $\mathcal{F}(z)$ of functions in $M(\mathcal{D}^2(\mathbb{D}))$ with $\max\{\text{rank } \mathcal{F}(z) \mid z \in \mathbb{D}\} = k \leq m$ and an $m \times 1$ vector of functions in $M(\mathcal{D}^2(\mathbb{D}))$ such that

- (i) $\det_k(\mathcal{F}(z)\mathcal{F}(z)^*)\alpha(\det_k(\mathcal{F}(z)\mathcal{F}(z)^*)) \geq |h_i(z)| \quad \forall z \in \mathbb{D}, i = 1, \dots, m$
- (ii) $\det_k(|\mathcal{F}'(z)\mathcal{F}(z)^*|)\alpha(\det_k(\mathcal{F}(z)\mathcal{F}(z)^*)) \geq |h'_i(z)| \quad \forall z \in \mathbb{D}, i = 1, \dots, m$
- (iii) $\|M_{\mathcal{F}}\| = 1$
- (iv) there exists a function $\underline{u} : \mathbb{D} \rightarrow l^2$ such that $\mathcal{F}\underline{u} = H$ everywhere on \mathbb{D}

then there exists an $\infty \times 1$ vector $\mathcal{G}(z)$ of functions in $M(\mathcal{D}^2(\mathbb{D}))$ such that

- (a) $\mathcal{F}(z)\mathcal{G}(z)^T = H(z) \quad \forall z \in \mathbb{D}$, and
- (b) $\|M_{\mathcal{G}}\| < \infty$.

Note that in this and other cases outside of $H^\infty(\mathbb{D})$ we would use Lemma 1 from [16] instead of (7) to estimate $\|M_G\|$.

4.3. Radicals. As previously noted, Wolff's condition (2) is not sufficient to show $h \in \mathcal{I}$. However, it is necessary *and* sufficient to show that h is contained in the *radical* of \mathcal{I} .

We would like to show a similar result for the matrix case. Let F and H be as before, with $\det_k(F(z)F(z)^*) \geq |h_i(z)|^n \forall z \in \mathbb{D}, i = 1, \dots, m$, and for some $n \in \mathbb{N}$, where $k = \max\{\text{rank } F(z) | z \in \mathbb{D}\}$. Suppose also that $\|M_F\| = 1$ and that we can find a \underline{u} such that $F\underline{u} = H$ on \mathbb{D} , as before. Then by Wolff's Theorem for Matrices, there exists an $\infty \times 1$ -vector G with entries in $H^\infty(\mathbb{D})$ such that $FG = H^{3n}$ everywhere on \mathbb{D} . (Here H^n is the vector obtained by raising each entry of H to the n th power.)

On the other hand, suppose we have $FG = H^n$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned} FG(G^*F^*) &= H^n(H^n)^* \Rightarrow \\ F\|M_G\|^2F^* &\geq H^n(H^n)^* \Rightarrow \\ \det_1(\|M_G\|^2FF^*) &\geq \det_1(H^n(H^n)^*) = \sum_{i=1}^m |h_i|^{2n} \Rightarrow \\ C \cdot \det_1(FF^*) &\geq |h_i|^{2n} \end{aligned}$$

$\forall z \in \mathbb{D}, i = 1, \dots, m$, where $C = \|M_G\|^{2m}$.

Note that if $k = \max\{\text{rank } F(z) | z \in \mathbb{D}\} = 1$, then this second statement is the converse of the first. It is currently unknown whether the converse holds for $k > 1$.

4.4. When H Is a Matrix. Our theorem extends easily to the case where H is an $m \times n$ matrix. If F is $m \times \infty$, we seek an $\infty \times n$ matrix G such that $FG = H$. Wolff's Theorem for Matrices allows us to find G by finding its n columns g_1, \dots, g_n .

What if we wish to solve an equation involving two (or more) matrices F_1 and F_2 ? That is, we wish to find G_1 and G_2 such that

$$F_1G_1 + F_2G_2 = H.$$

This is handled easily if we define $\mathcal{F} = [F_1 \ F_2]$; that is, \mathcal{F} is obtained by concatenating F_1 with F_2 (rearranging the entries in the case where F_1 and F_2 are $m \times \infty$). Then, provided hypotheses (i), (ii), and (iii) of our main theorem hold on \mathcal{F} , there exists \mathcal{G} such that

$$\mathcal{F}\mathcal{G} = H,$$

and from \mathcal{G} we obtain G_1 and G_2 such that

$$\mathcal{F}\mathcal{G} = F_1G_1 + F_2G_2.$$

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